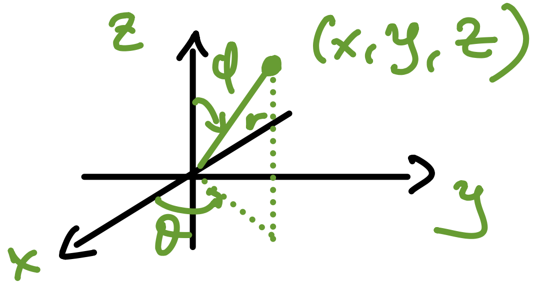
Info : La fin du programme est proche
→ révisions en regardant des questions d'exam
entre 2015 - 2021

Rappel : Coordonnées sphériques

$$(x,y,z) = (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)$$

$r \geq 0$, $\theta \in [\theta_0, \theta_0 + 2\pi]$, $\varphi \in [0, \pi]$

Jacobien $= r^2 \sin\varphi$



Exemple 7.19 (suite) $\to$ sphérique.

(iii) $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \leq 9,\ z \geq \sqrt{x^2+y^2}\}$

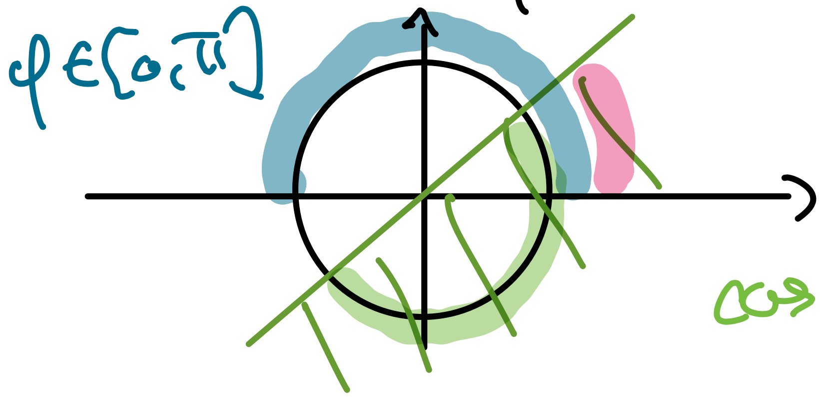
$(x,y,z) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$ $r\geq 0$, $\theta\in[\quad]$

$\varphi\in[0,\pi]$

$x^2+y^2+z^2 \leq 9 \overset{r\geq 0}{\Longleftrightarrow} 0\leq r\leq 3$

$z\geq\sqrt{x^2+y^2} \Longleftrightarrow r\cos\varphi \geq \sqrt{r^2\sin^2\varphi} = |r\sin\varphi| \overset{\substack{r\geq 0\\ \varphi\in[0,\pi]}}{=} r\sin\varphi$

$\longrightarrow \cos\varphi \geq \sin\varphi$



$\Rightarrow \varphi\in[0,\pi/4]$

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_0^3\int_0^{2\pi}\int_0^{\pi/4} f(-)\,r^2\sin\varphi\,d\varphi\,d\theta\,dr$$

$$y \geq \sqrt{x^2+z^2}$$

$$r\sin\theta\sin\varphi \geq \sqrt{r^2\cos^2\theta\sin^2\varphi + r^2\cos^2\varphi}$$

<u>Exemple 7.20</u> Coordonnées cylindriques.

Soit $G : \mathbb{R}_+ \times [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ défini par

peut être remplacé par n'importe quel intervalle de longueur 2π

$$G(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$$

$$\nabla G(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\operatorname{Jac} G(r,\theta,z) = \det \nabla G(r,\theta,z) = r$$

Exemples 7.21

(i) Volume du cylindre de rayon R et de hauteur h.

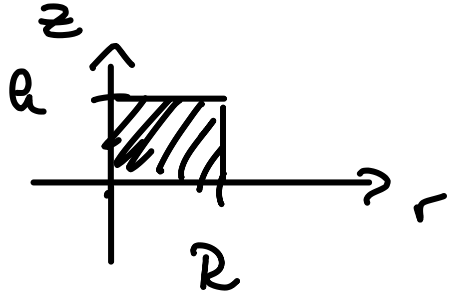
$$E = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq R^2 ,\quad 0 \leq z \leq h\}.$$

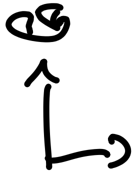
$$(x,y,z) = (r\cos\theta, r\sin\theta, z) \quad r \geq 0, \theta \in [\quad], z \in \mathbb{R}.$$

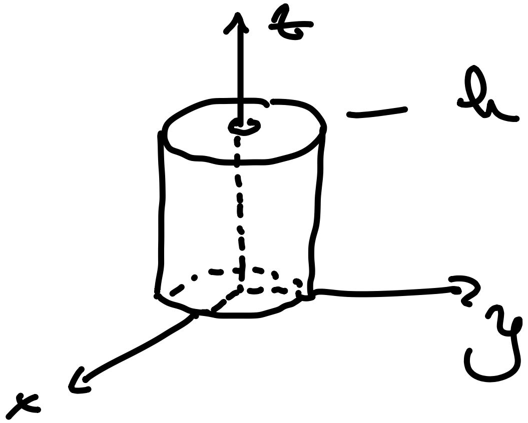
$x^2+y^2 \le R^2 \iff 0 \le r \le R$

$r \ge 0$

$0 \le z \le h \iff 0 \le z \le h$.







$$\mathrm{Vol}(E) = \int_0^{2\pi}\int_0^{R}\int_0^{h} 1 \quad r \; dz\,dr\,d\theta$$

Jacobien !

$$= \dots = \pi R^2 h$$

(ii) Ensemble simple dans le plan r, z + conditions sur θ

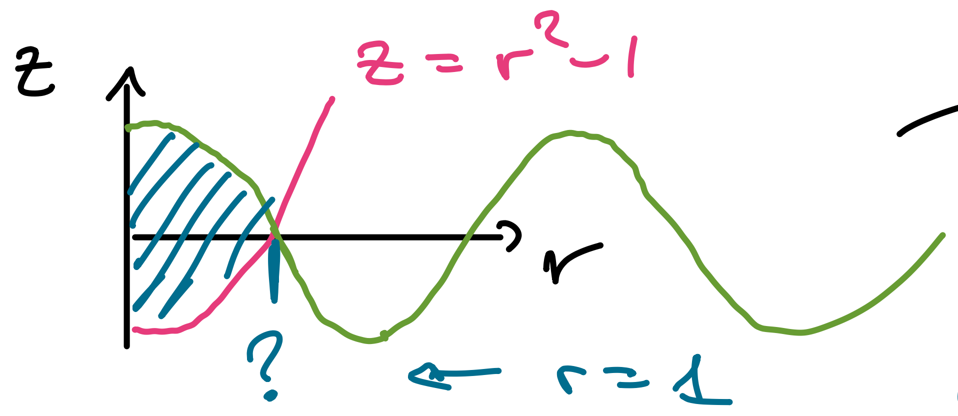
$$E = \left\{ (x,y,z) \in \mathbb{R}^3 : x \geq 0, \quad x^2+y^2-1 \leq z \leq \cos\left(\frac{\pi}{2}\sqrt{x^2+y^2}\right) \right\}$$

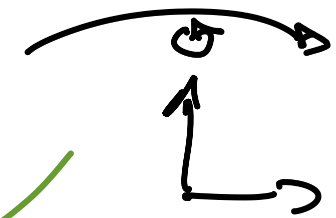
$(x,y,z) = (r\cos\theta, r\sin\theta, z) \quad r \geq 0, \quad \theta \in [-\pi, \pi], \quad z \in \mathbb{R}.$

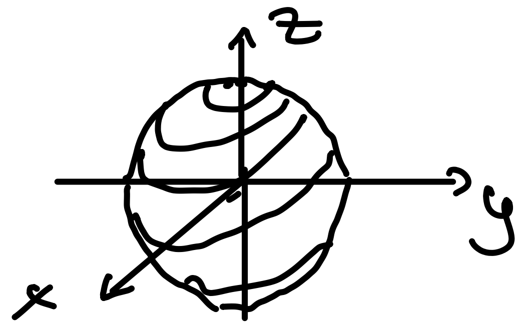
$x \geq 0 \iff r\cos\theta \geq 0 \iff \cos\theta \geq 0 \iff \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

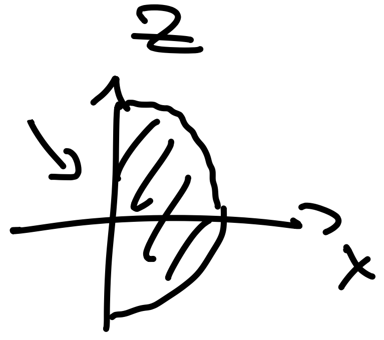
$$x^2+y^2-1 \leq z \leq \cos\left(\frac{\pi}{2}\sqrt{x^2+y^2}\right)$$

$$\iff r^2-1 \leq z \leq \cos\left(\frac{\pi}{2} r\right)$$

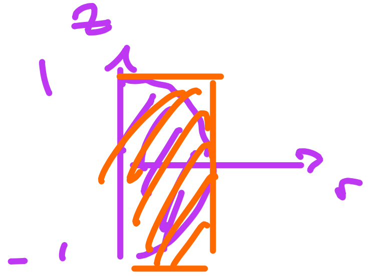








$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{r^2-1}^{\cos(\frac{\pi}{2}r)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, d\theta\, dr$$



$r \in [0,1]$, $z \in \left[r^2-1, \cos\left(\frac{\pi}{2}r\right)\right]$

$\Updownarrow$

$z \in [-1, 0]$ $\quad 0 \le r \le \sqrt{1+z}$

$z \in [0, 1]$ $\quad 0 \le r \le \frac{2}{\pi}\arccos(z)$

$z = r^2 - 1$

$z+1 = r^2$

$z = \cos\left(\frac{\pi}{2}r\right)$

(iii) Mélange cylindrique & cartésiennes

$x+y-6 \leq z \leq 6-x-y$

$$E = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 3,\ x \geq 0,\ |z| \leq 6-x-y \}$$

coordonnées cylindriques

↳ coordonnées cartésiennes.

En coordonnées cartésiennes :

z : $\boxed{x+y-6 \leq z \leq 6-x-y}$ $\to x+y-6 \leq 6-x-y \Leftrightarrow 6-x-y \geq 0$

Reste : $x^2+y^2 \leq 3$, $x \geq 0$, $6-x-y \geq 0$

$\to -\sqrt{3} \leq y \leq \sqrt{3}$

x : $x \geq 0$, $x \leq 6-y$, $x^2+y^2 \leq 3 \Leftrightarrow -\sqrt{3-y^2} \leq x \leq \sqrt{3-y^2}$

$$\max\left(0, -\sqrt{3-y^2}\right) \leq x \leq \min\left(6-y, \sqrt{3-y^2}\right)$$

$-\sqrt{3-y^2} \leq 0 \implies \max(0, -\sqrt{3-y^2}) = 0.$

$\sqrt{3-y^2} = 6-y \implies 3-y^2 = 36-12y+y^2$

$\iff 2y^2-12y+33=0$

$y = \frac{12 \pm \sqrt{12^2-4\cdot2\cdot33}}{4} = \frac{12\pm\sqrt{144-264}}{4}$

pas de solution $\Rightarrow$ toujours la même fonction réalise le minimum. en testant avec $y=0$

$\min(6-y, \sqrt{3-y^2}) = \sqrt{3-y^2}$

$\implies \boxed{0 \leq x \leq \sqrt{3-y^2}}$

$y : -\sqrt{3} \leq y \leq \sqrt{3}$.

$$\Rightarrow \iiint_E f(x,y,z)\,dx\,dy\,dz = \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} \int_{x+y-6}^{6-x-y} f(x,y,z)\,dz\,dx\,dy$$

Coordonnées cylindriques :

$(x,y,z) = (r\cos\theta, r\sin\theta, z)$ $r \geq 0$, $\theta \in [-\pi, \pi]$, $z \in \mathbb{R}$

$x^2+y^2 \leq 3 \Leftrightarrow 0 \leq r \leq \sqrt{3}$

$x \geq 0 \Leftrightarrow \theta \in [-\pi/2, \pi/2]$

$x+y-6 \leq z \leq 6-x-y \Leftrightarrow r(\cos\theta + \sin\theta) - 6 \leq z \leq 6 - r(\cos\theta + \sin\theta)$

$6 - r(\cos\theta + \sin\theta) \geq 0 \Leftarrow$

$r \in [0, \sqrt{3}]$, $6 - r\cos\theta - r\sin\theta \geq 6 - \sqrt{3} - \sqrt{3} \geq 0$.

$$\Rightarrow \iiint_E f(x,y,z)\,dx\,dy\,dz = \int_0^{\sqrt{3}} \int_{-\pi/2}^{\pi/2} \int_{r(\cos\theta+\sin\theta)-6}^{6-r(\cos\theta+\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,d\theta\,dr$$

(iv) Mélange sphérique & cylindrique :

$$E = \{(x,y,z) \in \mathbb{R}^3 ;\ x^2+y^2+z^2 \leq 9,\ z \geq \sqrt{x^2+y^2}\}$$

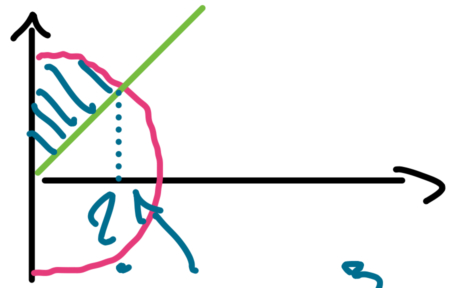
Déjà traité en coordonnées sphériques

En cylindriques :

$(x,y,z) = (r\cos\theta, r\sin\theta, z)$ $r \geq 0$, $\theta \in [\ \ ]$, $z \in \mathbb{R}$.

$$x^2+y^2+z^2 \leq 9 \iff r^2+z^2 \leq 9$$

$$z \geq \sqrt{x^2+y^2} \iff z \geq r$$



$r^2 + z^2 = 9$ & $z = r$, $r = z = \frac{3}{\sqrt{2}}$

$$r^2 + z^2 \leq 9 \iff -\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2}$$

$$\Rightarrow \iiint_E f(x,y,z)\,dx\,dy\,dz = \int_0^{3/\sqrt{2}} \int_0^{2\pi} \int_r^{\sqrt{9-r^2}} f(r\cos\theta, r\sin\theta, z)\cdot r \, dz\,d\theta\,dr.$$

__Remarque 7.22__

Le choix de garder z comme variable dans $\mathbb{R}$ est arbitraire.

$$x^2+y^2 \rightsquigarrow (r\cos\theta,\ r\sin\theta,\ z)$$

$$x^2+z^2 \rightsquigarrow (r\cos\theta,\ y,\ r\sin\theta)$$

$$y^2+z^2 \rightsquigarrow (x,\ r\cos\theta,\ r\sin\theta)$$